

Superactivation, unlockability, and secrecy distribution of bound information

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Bound information, a cryptographic classical analogue of bound entanglement, is defined as classical secret correlations from which no secret key can be extracted. Its existence was conjectured and shown in a multipartite case. In this work, we provide a new example of bound information in a four-partite scenario. Later, using this example, we prove that bound information can be superactivated in a finite-copy scenario and unlockable. We also show that bound entangled states (bound information) can be used to distribute multipartite pure-state entanglement (secret key).

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I. INTRODUCTION

Entanglement is one of the key resources that distinguish Quantum Information Theory from its classical counterpart. The impossibility for spatially separated parties to create entangled states by local operation and classical communication (LOCC) underlies the crucial role that entanglement plays for communication purposes. The property that entanglement is non-increasing under LOCC naturally leads to the definition of the maximally entangled state [1],

$$|\psi_{AB}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}.$$

This state is often called an entangled bit (ebit) and used as the unit of entanglement. Distilling ebits is a basic task in quantum communication scenarios.

The quantum distillation scenario shares many similarities with the information-theoretic secret key distillation scenario introduced in Ref. [2]. In this classical scenario, a secret bit (sbit) is aimed to be distilled from initially shared secret correlations using local operations and public communication (LOPC). An sbit is described by a probability distribution among honest parties Alice (A) and Bob (B) and an eavesdropper Eve (\mathcal{E}), $P_{AB\mathcal{E}}$, satisfying:

$$\begin{aligned} P_{AB\mathcal{E}}(a, b, e) &= P_{AB}(a, b) P_{\mathcal{E}}(e), \\ P_{AB}(a, b) &= \delta_{a, b}/2 \end{aligned} \quad (1)$$

for $a, b \in \{0, 1\}$. The former condition means that the honest parties are uncorrelated with Eve, and the latter shows perfect correlations between them. Analogous to properties of entanglement, secret correlations do not increase under LOPC [3]. Therefore, an sbit represents the maximal secret correlations and is used as the unit of classical correlations.

One can then relate an ebit with an sbit and vice versa as the basic analogues of units of correlations. Based on this correspondence, further analogies between quantum and classical scenarios follow, e.g. LOPC as a classical analogue of LOCC [3]. Moreover, interrelations of information-theoretic quantities and their operational

meanings have also been investigated in [4] [5] [6].

There are also well-established quantitative relations between quantum and secret correlations. To introduce them, let us briefly remind two entanglement measures presented in Refs [7, 9]. Each measure has operational meaning, one for the formation and the other for the distillation of entangled quantum states. The entanglement cost denoted by E_c quantifies the number of ebits for the formation of a given quantum state ρ_{AB} in the asymptotic limit, i.e. when the number of copies of ρ_{AB} tends to be very large [7]. Entanglement cost is positive if and only if a given state is entangled [8]. In a similar way, the number of ebits that can be distilled out of given quantum states in the asymptotic limit is quantified and called entanglement of distillation denoted by E_D [9]. In general, it holds that $E_c \geq E_D$ as one cannot get more entanglement from given states than that used for the preparation. The existence of undistillable, i.e. bound, entangled states [10], shows the irreversibility in the entanglement manipulation and appears when $E_c > E_D = 0$. It has been shown that, despite their undistillability *per se*, bound entangled states can be activated [10] [11] [12].

Similar questions can be addressed in the context of the classical cryptographic scenario. That is, when a probability distribution $P_{AB\mathcal{E}}$ is given, the information of formation denoted by I_c has been derived as the classical analog of E_c [13], and determines the number of sbits for the formation of a given distribution via LOPC. Positive information of formation, $I_c(A : B|C) > 0$, means that a given probability distribution contains secret correlations. For the distillation, the natural classical analog is the secret key rate, denoted by $S(A : B|\mathcal{E})$ [14], the number of sbits that can be distilled from given classical correlations in the asymptotic limit.

Quantum and classical correlations can be related by measurement, i.e. $P_{AB\mathcal{E}} = \text{tr}[\rho_{AB\mathcal{E}} M_A \otimes M_B \otimes M_{\mathcal{E}}]$ where M_j for $j = A, B, \mathcal{E}$ are positive-operator-valued-measure on quantum state $\rho_{AB\mathcal{E}}$. It immediately follows that an sbit is obtained by measuring an ebit in the computational basis. Entanglement and secret correlations are also generally interrelated. That is, given quantum

states, if they are entangled, there always exist local measurements such that the measured outcomes contain secret correlations. As well, given probability distributions obtained by measuring quantum states, if they consist of secret correlations, then the corresponding quantum states must be entangled, i.e. $E_c > 0 \Leftrightarrow I_c > 0$ [15].

As a classical analog of bound entanglement, bound information was defined as secret correlations from which no sbit can be distilled [4]. The existence of bound information was then explored, and was indeed shown in Ref. [16]. An instance of bound information was explicitly provided in a multipartite scenario, and remarkably, was obtained by measuring bound entangled states presented in Ref. [12]. Moreover, it was shown that bound information can be activated in an asymptotic-copy scenario [16]. Note that this is also called superactivation in the sense that individual probability distributions consisting of bound information are undistillable while the one combined together can be converted to distillable correlations.

In this work, along the interrelation between quantum and secret correlations, we study further properties of bound information. We introduce the classical analogues of finite-copy superactivation and the unlockability of bound entanglement. All these findings are based on the intriguing properties of the Smolin state introduced in Ref. [17]. We also discover a useful feature of the undistillable correlations in distributing pure-state entanglement and multipartite sbits. In the quantum scenario, it is shown that the tripartite GHZ state can be extended to the four-partite GHZ state using LOCC, given that a four-partite bound entangled state is shared among parties. A classical analogue also follows. When bound information is shared by four parties, an sbit of three parties can be distributed over the four parties using LOPC. Note also that the Smolin state is immediately a feasible resource that has been implemented with present-day technology [18].

This paper is organized as follows. In Sec. II, the properties of Smolin states are briefly reviewed. In Sec. III, the bound information is then derived by measurement on the Smolin state, and the properties such as unlockability and superactivation are translated. In Sec. IV, it is shown that bound entangled state (bound information) together with LOCC (LOPC) can be used to extend GHZ states (sbits) from three to four parties.

II. THE SMOLIN STATE

Let us first briefly review the properties of the Smolin state presented in Ref. [17]. The Smolin state is a four-partite bound entangled state, shared by, say Alice, Bob, Clare and David:

$$\rho_{ABCD} = \frac{1}{4} \sum_i |\psi_i\rangle_{AB} \langle \psi_i| \otimes |\psi_i\rangle_{CD} \langle \psi_i|, \quad (2)$$

where $|\psi_1\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, $|\psi_2\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$, $|\psi_3\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$, and $|\psi_4\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. This state has been exploited to derive intriguing effects of bound entanglement such as the unlockability and the superactivation in a finite-copy scenario [17]. Let us summarize the properties in the following.

- (i) *Invariance under permutations.* The state is symmetric under any exchange of parties, i.e. $\rho_{ABCD} = \rho_{ABDC} = \rho_{ADBC}$.
- (ii) *Undistillability.* Looking at the bipartite splitting $AB : CD$ in the state in Eq. (2), it is clear that the state is separable across the cut. Then, from the property (i), it follows that the state is separable in all bipartitions across two parties versus the others, such as $AC : BD$ and $AD : BC$. This already shows that no pair of parties can distill entanglement, and therefore the state is undistillable.
- (iii) *Unlockability.* An important property of the state is the unlockability of entanglement. This can be seen when two parties among the four join together and apply collective operations to discriminate among the four Bell states. Announcing the measurement outcome, the two joined parties can allow the other two parties to know which Bell state is shared between them. Then, applying local unitaries that depend on the announced outcome, they can finally distill the Bell state $|\psi_1\rangle$. This shows that the Smolin state is entangled, and also bound entangled together with the property (ii).

Superactivation with finite copies. One of the intriguing effects in the entanglement theory is that bound entangled states can be activated. The Smolin state was in particular exploited to show a strong version of the effect, superactivation in a finite-copy scenario. It is called superactivation in the sense that $E_D(\rho_1 \otimes \rho_2) > 0$ while $E_D(\rho_1) = E_D(\rho_2) = 0$. This was shown in Ref. [19], and the activation works as follows. Suppose that, now including the fifth one Elena, two copies of the Smolin state are shared by the five parties in the following way,

$$\rho_{A_1 C_1 B_1 D} \otimes \rho_{A_2 B_2 C_2 E}, \quad (3)$$

where the first and the second copies are labeled. Then, David and Elena distill an ebit, applying the following protocol, see also Fig. 1. First, Alice teleports her qubit state of A_2 to Clare sacrificing the unknown Bell state shared between A_1 and C_1 . Clare is then with two qubits C'_1 and C_2 where C'_1 is in the teleported state from A_2 . Next, Bob teleports his qubit state of B_2 to D using the unknown Bell state shared between B_1 and D . Then, David is now with D' in the teleported state from B_2 . Finally, due to the structure of the Smolin state, the state $C'_1 C_2 D' E$ shared by Clare, David, and Elena results in the Smolin state. Since Clare holds two qubits C'_1 and C_2 , she can discriminate among Bell states and announces the result, by which David and Elena can distill an ebit.

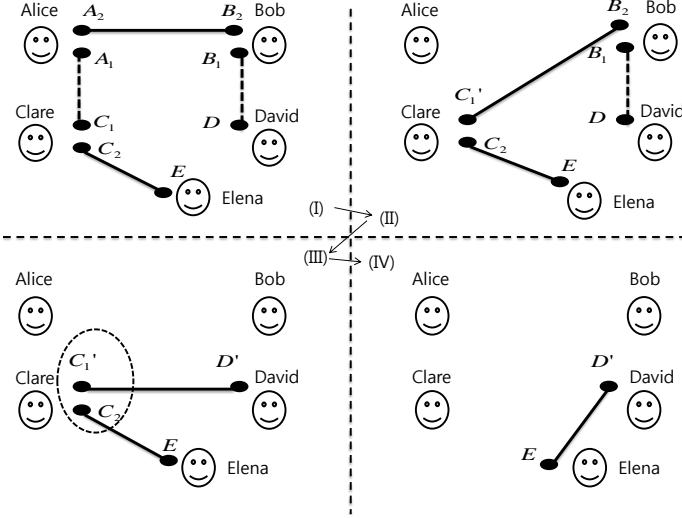


FIG. 1: The activation protocol for both the quantum and classical scenarios is shown. For the quantum scenario, two Smolin states $\rho_{A_1 C_1 B_1 D}$ and $\rho_{A_2 B_2 C_2 E}$ are drawn with the dashed and solid lines, respectively. For the classical scenario, the Smolin states are simply replaced with the bound information in (7). In both cases, the first step in the protocol (shown in I→II) is that Alice teleports her state in the system A_2 to C_1 using the correlation between $A_1 C_1$. In the second step (shown in II→III), Bob teleports the state of his system B_2 to D sacrificing the correlation existing in $B_1 D$. Then, the resulting distribution over the remaining three parties, Clare holding two systems, David, and Elena is in fact the Smolin state if the scenario is with quantum systems, or the bound information in (7) if it is with classical systems. Finally (shown in III→IV), Clare measures her systems and announces the outcomes, so that David and Elena distill an ebit or an sbit.

Now, symmetrizing the state in Eq. (3) with more copies as follows,

$$\rho_{ABCD} \otimes \rho_{ABCE} \otimes \rho_{ABDE} \otimes \rho_{ACDE} \otimes \rho_{BCDE}, \quad (4)$$

any two parties among the five can distill ebits. In this way, from ebits shared by every two parties, it follows that multipartite pure entangled states can be distilled. This finally shows that multipartite bound entangled states can be superactivated. It is also noteworthy in the activation scenario that, the distillable entanglement defined in the asymptotic limit becomes immediately positive in a finite number of copies.

III. BOUND INFORMATION

Bound information is characterized by two constraints on probability distributions, i) existence of secret correlations $I_c(A : B|\mathcal{E}) > 0$ and ii) the undistillability $S(A : B||\mathcal{E}) = 0$. There is an information measure located in the between, the intrinsic information

$I(A : B \downarrow \mathcal{E})$ [2], such that

$$I_c(A : B|\mathcal{E}) \geq I(A : B \downarrow \mathcal{E}) \geq S(A : B||\mathcal{E}). \quad (5)$$

Its usefulness lies on the fact that a general undistillability criterion is lacked and zero-valued intrinsic information is immediately a sufficient condition for the undistillability. This condition is therefore going to be used later on, when proving that a given probability distribution contains bound information. The intrinsic information is defined as the minimum of the conditional mutual information of honest parties given an eavesdropper,

$$I(A : B \downarrow \mathcal{E}) = \min_{\mathcal{E} \rightarrow \bar{\mathcal{E}}} I(A : B|\bar{\mathcal{E}}), \quad (6)$$

over all Eve's stochastic mappings $\mathcal{E} \rightarrow \bar{\mathcal{E}}$.

In the following subsections, we show that measurements on the Smolin states in the computational basis in fact give bound information. We then show that bound information can be superactivated in a finite-copy scenario, analogously to the quantum case.

A. Bound information and the unlockability

The entanglement properties can be related to cryptographic properties of the probability distributions that are obtained by measuring given quantum states. Without loss of generality, one can assume that Eve has access to the rest of legitimate parties, and this is expressed by the fact that Eve holds the purification. For instance, when the Smolin state ρ_{ABCD} is shared, one can find a state $|\psi\rangle_{ABCDE}$ such that $\rho_{ABCD} = \text{tr}_{\mathcal{E}}|\psi\rangle\langle\psi|_{ABCDE}$. In this way, Eve is naturally included and her correlations with the legitimate parties are readily shown. Denoted by positive operator M_α of party α , the probability distribution P_{ABCDE} of the five parties reads,

$$\text{tr}[M_A \otimes M_B \otimes M_C \otimes M_D \otimes M_{\mathcal{E}}|\psi\rangle\langle\psi|_{ABCDE}].$$

Suppose that measurements applied by the parties are in the computational basis. The probability distribution is explicitly given by,

| A | C | B | D | \mathcal{E} | P_{ACBDE} |
|-----|-----|-----|-----|---------------|-------------|
| 0 | 0 | 0 | 0 | ϵ_1 | 1/8 |
| 0 | 0 | 1 | 1 | ϵ_2 | 1/8 |
| 1 | 1 | 0 | 0 | ϵ_2 | 1/8 |
| 1 | 1 | 1 | 1 | ϵ_1 | 1/8 |
| 0 | 1 | 0 | 1 | ϵ_3 | 1/8 |
| 0 | 1 | 1 | 0 | ϵ_4 | 1/8 |
| 1 | 0 | 0 | 1 | ϵ_4 | 1/8 |
| 1 | 0 | 1 | 0 | ϵ_3 | 1/8 |

In what follows, we show that, analogously to the quantum case, the distribution (7) in fact contains bound information which is also unlockable. All these are obtained as classical analogues of the properties shown in the Smolin state in Eq. (2).

- (i') *Invariance under permutations.* The distribution (7) is invariant under permutations of parties, i.e. $P_{ACBDE} = P_{ABCDE} = P_{ADBCE}$.
- (ii') *Undistillability.* The distribution $P(A, C, B, D, \mathcal{E})$ is undistillable in every bipartition across two parties versus the others. That is, for instance in the bipartition between AC and BD , it holds that

$$I(AC : BD \downarrow \mathcal{E}) = 0, \quad (8)$$

where Eve's local mapping is given by, $\epsilon_2 \rightarrow \epsilon_1$ and $\epsilon_3 \rightarrow \epsilon_4$. From the relation in (5), it follows that $S(AC : BD \parallel \mathcal{E}) = 0$. Then, the permutational invariance in (i') implies $S(AB : CD \parallel \mathcal{E}) = S(AD : BC \parallel \mathcal{E}) = 0$, and therefore none of two parties can distill an sbit.

- (iii') *Unlockability.* The secret correlations existing in (7) are unlockable. Suppose two parties, for instance B and D , join together and post-select either case that they are the same or different. Let us now restrict to the case that B and D accept when they share the same bit values. Then, the distribution is given by

| A | C | B | D | \mathcal{E} | P_{ACBDE} |
|-----|-----|-----|-----|---------------|-------------|
| 0 | 0 | 0 | 0 | ϵ_1 | $1/4$ |
| 0 | 0 | 1 | 1 | ϵ_2 | $1/4$ |
| 1 | 1 | 0 | 0 | ϵ_2 | $1/4$ |
| 1 | 1 | 1 | 1 | ϵ_1 | $1/4$ |

(9)

This means that an sbit is distilled between A and C , since it is clear in the distribution (9) that i) $P_{AC}(0,0) = P_{AC}(1,1) = 1/2$ and ii) $P_{AC\mathcal{E}}(a,c,e) = P_{AC}(a,c)P_{\mathcal{E}}(e)$. For the other case that B and D accept whenever they share different bit values, applying the bit-flip operation either A and C , Alice and Clare can distill an sbit. From the symmetry property in (i'), it immediately follows that any two parties who join and collaborate to identify the shared state can allow the other two parties to distill an sbit. As an sbit is distilled, this also means that the probability distribution in (7) consists of secret correlations. Together with the undistillability in (ii'), it is shown that the distribution in (7) indeed contains bound information.

B. Superactivation

In this subsection, we show that bound information can be superactivated in a finite-copy scenario. We first show that an sbit can be distilled by two parties when two copies of the bound information in Eq. (7) are shared by five parties. Then, it follows with more copies that multipartite sbits are distilled by the five parties.

Let us begin with the the following probability distribution shared by the five parties,

$$P_{ABCDE} = P_{A_1C_1B_1D}P_{A_2B_2C_2E}, \quad (10)$$

where each four-partite distribution is shown in Eq. (7) and the first and the second copies are labeled. Note that the distribution in Eq. (10) can also be obtained by directly measuring the tensored state in (3) in the computational basis. To be explicit, the distribution in Eq. (10) shows, for $i, j = 0, 1$,

| A_1 | A_2 | B_1 | B_2 | C_1 | C_2 | D | E | \mathcal{E}_1 | \mathcal{E}_2 |
|-------|-------|-------|-------|-------|-------|-------|-------|-----------------|-----------------|
| i | j | i | j | i | j | i | j | ϵ_1 | f_1 |
| i | j | i | j | i | $j+1$ | i | $j+1$ | ϵ_1 | f_2 |
| i | j | i | $j+1$ | i | j | i | $j+1$ | ϵ_1 | f_3 |
| i | j | i | $j+1$ | i | $j+1$ | i | j | ϵ_1 | f_4 |
| i | j | $i+1$ | j | i | j | $i+1$ | j | ϵ_2 | f_1 |
| i | j | $i+1$ | j | i | $j+1$ | $i+1$ | $j+1$ | ϵ_2 | f_2 |
| i | j | $i+1$ | $j+1$ | i | j | $i+1$ | $j+1$ | ϵ_2 | f_3 |
| i | j | $i+1$ | $j+1$ | i | $j+1$ | $i+1$ | j | ϵ_2 | f_4 |
| i | j | i | j | $i+1$ | j | $i+1$ | j | ϵ_3 | f_1 |
| i | j | i | j | $i+1$ | $j+1$ | $i+1$ | $j+1$ | ϵ_3 | f_2 |
| i | j | i | $j+1$ | $i+1$ | j | $i+1$ | $j+1$ | ϵ_3 | f_3 |
| i | j | i | $j+1$ | $i+1$ | $j+1$ | $i+1$ | j | ϵ_3 | f_4 |
| i | j | $i+1$ | j | $i+1$ | j | i | j | ϵ_4 | f_1 |
| i | j | $i+1$ | j | $i+1$ | $j+1$ | i | $j+1$ | ϵ_4 | f_2 |
| i | j | $i+1$ | $j+1$ | $i+1$ | j | i | $j+1$ | ϵ_4 | f_3 |
| i | j | $i+1$ | $j+1$ | $i+1$ | $j+1$ | i | j | ϵ_4 | f_4 |

(11)

This expression can be obtained using a simpler form of Eq. (7) that is shown in the appendix A.

The classical analogue of the quantum teleportation, which is to be used in the activation protocol, is in fact the one-time pad that securely sends a classical bit by sacrificing an sbit. For convenience, we also call this "teleporting" classical bits, which works as follows. Assume that an sbit s , is shared by two honest parties. The sender encodes a message x and publicly announces the addition $(x+s)$, so that the receiver can decode the message by adding the shared sbit, $(x+s) + s$. Since the value of the sbit s is not known to anyone else, one can only guess a random bit from the public communication.

The activation protocol is obtained by translating the quantum one, and works as follows, see also Fig. 1. First, Alice teleports her bit in A_2 to Clare, sacrificing an sbit in A_1C_1 . Clare then has a new value in the register, $C'_1 = C_1 + A_1 + A_2$, and the probability distribution

becomes as follows,

| B_1 | B_2 | C'_1 | C_2 | D | E | \mathcal{E}_1 | \mathcal{E}_2 |
|-------|-------|--------|-------|-------|-------|-----------------|-----------------|
| i | j | j | j | i | j | ϵ_1 | f_1 |
| i | j | j | $j+1$ | i | $j+1$ | ϵ_1 | f_2 |
| i | $j+1$ | j | j | i | $j+1$ | ϵ_1 | f_3 |
| i | $j+1$ | j | $j+1$ | i | j | ϵ_1 | f_4 |
| $i+1$ | j | j | j | $i+1$ | j | ϵ_2 | f_1 |
| $i+1$ | j | j | $j+1$ | $i+1$ | $j+1$ | ϵ_2 | f_2 |
| $i+1$ | $j+1$ | j | j | $i+1$ | $j+1$ | ϵ_2 | f_3 |
| $i+1$ | $j+1$ | j | $j+1$ | $i+1$ | j | ϵ_2 | f_4 |
| i | j | $j+1$ | j | $i+1$ | j | ϵ_3 | f_1 |
| i | j | $j+1$ | $j+1$ | $i+1$ | $j+1$ | ϵ_3 | f_2 |
| i | $j+1$ | $j+1$ | j | $i+1$ | $j+1$ | ϵ_3 | f_3 |
| i | $j+1$ | $j+1$ | $j+1$ | $i+1$ | j | ϵ_3 | f_4 |
| $i+1$ | j | $j+1$ | j | i | j | ϵ_4 | f_1 |
| $i+1$ | j | $j+1$ | $j+1$ | i | $j+1$ | ϵ_4 | f_2 |
| $i+1$ | $j+1$ | $j+1$ | j | i | $j+1$ | ϵ_4 | f_3 |
| $i+1$ | $j+1$ | $j+1$ | $j+1$ | i | j | ϵ_4 | f_4 |

Next, Bob teleports his value in B_2 to David sacrificing an sbit in B_2D . Then, David holds a new value $D' = D + B_1 + B_2$, with which the probability distribution of the four parties is given by

| C'_1 | C_2 | D' | E | \mathcal{E}_1 | \mathcal{E}_2 |
|--------|-------|-------|-------|-----------------|-----------------|
| j | j | j | j | ϵ_m | f_1 |
| j | $j+1$ | j | $j+1$ | ϵ_m | f_2 |
| j | j | $j+1$ | $j+1$ | ϵ_m | f_3 |
| j | $j+1$ | $j+1$ | j | ϵ_m | f_4 |

where $m = 1, 2, 3, 4$. The explicit form of the distribution of Eq. (13) is shown in the appendix B. Now, the distribution in Eq. (13) is identical to the bound information in Eq. (7). Remind that the secret correlations in (7) is unlockable, as it was shown in Sec.III A. Therefore, Clare, who is with two bits C'_1 and C_2 , announces if her two values are the same or not, depending on which, by applying local operations David and Elena can share an sbit: if it is announced that C'_1 and C_2 are unequal, either David or Elena applies the bit-flip operation. It is therefore shown that an sbit can be distilled between D and E .

Moreover, symmetrizing the distribution in Eq. (10), i.e.,

$$P_{ABCDE_1}P_{ABCE_2}P_{ABDE_3}P_{ACDE_4}P_{BCDE_5}, \quad (14)$$

any two parties among the five can distill sbits against an eavesdropper who holds the five random variables $\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4\mathcal{E}_5$. Therefore, it is straightforward that, with more copies, the five parties can share secrecy.

IV. DISTRIBUTION OF ENTANGLEMENT AND SECRECY

In this section, we show a usefulness of undistillable correlations in quantum and classical scenarios, respec-

tively, namely that they can be used to distribute multipartite distillable correlations. In the quantum scenario, we consider distribution of multi-partite GHZ state,

$$|\phi_N\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}.$$

We show that tripartite GHZ state can be deterministically extended into four parties using LOCC when the Smolin state is shared by the four parties.

We also derive a classical analogue of the quantum state distribution. Multipartite sbits of N parties, say A_1, \dots, A_N , is a classical analogue of the N -partite GHZ state, being defined as the following probability distribution

$$P_{A_1, \dots, A_N}(a_1, \dots, a_N) = \delta_{a_1, a_2} \delta_{a_2, a_3} \dots \delta_{a_{N-1}, a_N} / 2,$$

$$P_{A_1, \dots, A_N, \mathcal{E}}(a_1, \dots, e) = P_{A_1, \dots, A_N}(a_1, \dots, a_N) P_{\mathcal{E}}(e).$$

We then show that the tripartite sbit can be extended into four parties using LOPC when the bound information in Eq. (7) is shared by the four parties. Note that in both quantum and classical scenarios the distribution scheme works deterministically.

A. Quantum scenario

Suppose that Alice, Bob, Clare, and David share the Smolin state, and that only three of them, say Alice, Bob, and Clare, additionally share a tripartite GHZ state as follows

$$\mu_{ABCD} = |\eta\rangle\langle\eta|_{ABCD} \otimes \rho_{ABCD} \quad (15)$$

where $|\eta\rangle_{ABCD} = |\phi_3\rangle_{ABC} \otimes |+\rangle_D$ and $|+\rangle_D = (|0\rangle + |1\rangle)/\sqrt{2}$. Let Λ_α for $\alpha = A, B, C, D$ denote the local operation performed by the party α . The goal is now to show that the state μ_{ABCD} can be transformed to $|\phi_4\rangle$ using some local operations Λ_α . To this end, the local operation, $\Lambda_\alpha : \mathbb{C}^2 \otimes \mathbb{C}^2 \rightarrow \mathbb{C}^2$, mapping from two-qubit to a single qubit states, can be explicitly constructed in terms of the Kraus operators, $K_0^\alpha = |0\rangle\langle 00| + |1\rangle\langle 11|$ and $K_1^\alpha = |0\rangle\langle 01| + |1\rangle\langle 10|$ as follows

$$\Lambda_\alpha(\cdot) = \sum_{i=0,1} K_i^\alpha(\cdot) K_i^{\alpha\dagger}. \quad (16)$$

The above can be in fact rephrased as collective measurement by which it can only be known if two qubit systems are in the same state or not, leaving a single qubit system.

Now, the four parties apply the local operation (16) to the state in (15). Suppose that four parties get measurement outcomes (i_A, j_B, k_C, l_D) . This happens with probability

$$\text{tr}[\mu_{ABCD} K_{i_A}^{A\dagger} K_{i_A}^A \otimes K_{j_B}^{B\dagger} K_{j_B}^B \otimes K_{k_C}^{C\dagger} K_{k_C}^C \otimes K_{l_D}^{D\dagger} K_{l_D}^D],$$

and the state resulted in the four parties is,

$$|\phi^v\rangle = \mathbf{1}_A \otimes \mathbf{1}_B \otimes \mathbf{1}_C \otimes (\sigma_D^x)^v |\phi_4\rangle, \quad (17)$$

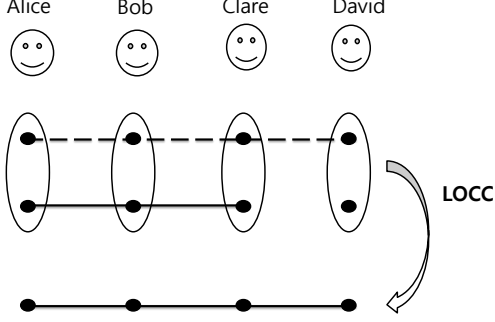


FIG. 2: Distribution of multipartite pure entanglement and secret key: Four parties share the Smolin states (dashed line) and are allowed to apply LOCC. Then, the tripartite GHZ state (solid line) can be distributed over the four parties using LOCC. The classical analogue also follows: the tripartite sbit can be distributed over the four using LOPC when the four-partite bound information in Eq. (7) is shared.

where $v = i_A + j_B + k_B + l_D$ and σ_D^x denotes the Pauli matrix σ_x in the David's side. Therefore, using classical communication to discuss the measurement outcomes, the four parties can compute, $v = i_A + j_B + k_C + l_D$. If v is an even number, this means that the four-partite GHZ state is already shared. Otherwise, David applies the σ_x operation to his qubit, and the four-partite GHZ state can be shared.

B. Classical scenario

Suppose that the four parties share the bound information in (7) and that only three of them, say Alice, Bob, and Clare, share an sbit in unknown value s for $s \in \{0, 1\}$. Let i_k for $k = A, B, C, D$ denote the value of the party k . The goal is then to distribute the tripartite sbit into the four parties using LOPC, such that David also securely share the sbit with the rest.

The distribution protocol works as follows, see also Fig. 2. Each of the three parties sharing the sbit, individually and locally copy the sbit and then compute the parity of the two bits, one from the sbit s and the other from the bound information i_k for $k = A, B, C$. Then, using the distribution in (21) which is a simpler expression of (7), the distribution is transformed as follows,

| A | C | B | D | \mathcal{E} | P_{ABCDE} |
|---------|-------------|-------------|---------|---------------|-------------|
| $s + i$ | $s + i$ | $s + i$ | i | ϵ_1 | $1/8$ |
| $s + i$ | $s + i$ | $s + i + 1$ | $i + 1$ | ϵ_2 | $1/8$ |
| $s + i$ | $s + i + 1$ | $s + i$ | $i + 1$ | ϵ_3 | $1/8$ |
| $s + i$ | $s + i + 1$ | $s + i + 1$ | i | ϵ_4 | $1/8$ |

Afterwards, each party publicly announces the parity bit $s + i_k$, so that David knows them and computes

the sum of the announced bit values, denoted by $v_D = \sum_{k=A,B,C} (s + i_k)$. David then adds v_D to his bit i_D as follows,

| A | C | B | D | \mathcal{E} | P_{ABCDE} |
|---------|-------------|-------------|---------------|---------------|-------------|
| $s + i$ | $s + i$ | $s + i$ | $i + v_D$ | ϵ_1 | $1/8$ |
| $s + i$ | $s + i$ | $s + i + 1$ | $i + 1 + v_D$ | ϵ_2 | $1/8$ |
| $s + i$ | $s + i + 1$ | $s + i$ | $i + 1 + v_D$ | ϵ_3 | $1/8$ |
| $s + i$ | $s + i + 1$ | $s + i + 1$ | $i + v_D$ | ϵ_4 | $1/8$ |

(19)

Eve has classical bits that are correlated with the four parties as it is shown in Eq. (7), and also listens to the announced bit values $s + i_k$ from the public communication. Then, as it is shown in Eq. (18), Eve can only discriminates among the four possibilities of ϵ_i for $i = 1, 2, 3, 4$. However, the sbit s shared by the three parties has not been known to Eve, who can make a random guess.

As it is shown in (19), the bit value of David results in, and is explicitly computed as

$$i_D + \sum_{k=A,B,C} s + i_k = s. \quad (20)$$

This is because, from the distribution of the bound information (7), it holds that $\sum_{k=A,B,C,D} i_k = 0$. Hence, it is shown that a multipartite sbit can be distributed securely via bound information together with LOPC.

V. CONCLUSION

We have shown a case of four-partite bound information and its properties, unlockability and superactivation. All these are obtained by deriving classical analogues of the Smolin state and its quantum effects, superactivation and unlockability in bound entangled states. It would be interesting to investigate which properties of quantum correlations can or cannot have their classical counterparts. For instance, existence of bipartite bound information remains open and is an challenging issue. Finally, we have shown a usefulness of undistillable correlations: bound entanglement and bound information can be used to distribute a multipartite GHZ state and multipartite sbits in quantum and classical scenarios, respectively.

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Appendix A: Derivation of (11)

By individual measurement to each copy of two Smolin states in (3), the five parties share measurement data such that Alice, Bob, and Clare possess two values labeled 1 and 2 and David and Elena keep single values. Both the first and the second distributions in the form in (7) can be written in a simpler form as follows. For the first copy,

| A_1 | C_1 | B_1 | D_1 | \mathcal{E}_1 | $P_{A_1 B_1 C_1 D \mathcal{E}}$ |
|-------|-------|-------|-------|-----------------|---------------------------------|
| i | i | i | i | ϵ_1 | $1/8$ |
| i | i | $i+1$ | $i+1$ | ϵ_2 | $1/8$ |
| i | $i+1$ | i | $i+1$ | ϵ_3 | $1/8$ |
| i | $i+1$ | $i+1$ | i | ϵ_4 | $1/8$ |

(21)

where $i = 0, 1$, and for the second copy of A_2 , B_2 , C_2 and E , assuming Eve holding the second parameter f_k , $k = 1, 2, 3, 4$,

| A_2 | B_2 | C_2 | E | \mathcal{E}_2 | $P_{A_2 B_2 C_2 D \mathcal{E}}$ |
|-------|-------|-------|-------|-----------------|---------------------------------|
| j | j | j | j | f_1 | $1/8$ |
| j | j | $j+1$ | $j+1$ | f_2 | $1/8$ |
| j | $j+1$ | j | $j+1$ | f_3 | $1/8$ |
| j | $j+1$ | $j+1$ | j | f_4 | $1/8$ |

(22)

for $j = 1, 2$. The full probability obtained by measuring the state in (3) is then shown in (11).

Appendix B: The full distribution of (13)

The full distribution of (13) is explicitly shown as follows, for different values of \mathcal{E}_1 ,

| C'_1 | C_2 | D' | E | \mathcal{E}_1 | \mathcal{E}_2 |
|--------|-------|-------|-------|-----------------|-----------------|
| j | j | j | j | ϵ_1 | f_1 |
| j | $j+1$ | j | $j+1$ | ϵ_1 | f_2 |
| j | j | $j+1$ | $j+1$ | ϵ_1 | f_3 |
| j | $j+1$ | $j+1$ | j | ϵ_1 | f_4 |
| j | j | j | j | ϵ_2 | f_1 |
| j | $j+1$ | j | $j+1$ | ϵ_2 | f_2 |
| j | j | $j+1$ | $j+1$ | ϵ_2 | f_3 |
| j | $j+1$ | $j+1$ | j | ϵ_2 | f_4 |
| $j+1$ | j | $j+1$ | j | ϵ_3 | f_1 |
| $j+1$ | $j+1$ | $j+1$ | $j+1$ | ϵ_3 | f_2 |
| $j+1$ | j | j | $j+1$ | ϵ_3 | f_3 |
| $j+1$ | $j+1$ | j | j | ϵ_3 | f_4 |
| $j+1$ | j | $j+1$ | j | ϵ_4 | f_1 |
| $j+1$ | $j+1$ | $j+1$ | $j+1$ | ϵ_4 | f_2 |
| $j+1$ | j | j | $j+1$ | ϵ_4 | f_3 |
| $j+1$ | $j+1$ | j | j | ϵ_4 | f_4 |

(23)

For cases when Eve is with ϵ_3 or ϵ_4 , the distribution in (13) can be obtained by replacing j with $j+1$ in (23).

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